Int J Theor Phys (2010) 49: 1628–1632
DOI 10.1007/s10773-010-0345-y

Quantum-like Interpretation of Dirac Wave Equation in Robertson-Walker Geometry

Antonio Zecca

Received: 1 February 2010 / Accepted: 31 March 2010 / Published online: 9 April 2010

Abstract The Dirac wave equation is separated in the Robertson-Walker metric. The resulting radial equation is interpreted as a one dimensional quantum-like equation that is explicitly solved. There results that the energy spectrum, that is determined in the flat, open and closed universe, is independent of the mass of the particle. Moreover it is the same of the massless neutrino case previously studied. In the closed metric case the discrete positive spectrum is asymptotically determined. The separation of the energy levels is however very far from being experimentally tested.

Keywords Robertson-Walker metric · Dirac equation · Solution · Energy spectrum

1 Introduction

The Dirac equation in Robertson-Walker (RW) space-time has been integrated in different ways by variable separation (e.g., [1–4]). What seems to lack, except for the massless neutrino case [5], is the reduction of the Dirac equation to a one dimensional wave equation, in analogy to what done in the case of the Kerr metric [6]. It is the object of the present paper to supply such formulation. To that end a previous study of separation of Dirac equation in RW metric [4] is reconsidered. According to it the separated angular equation is integrated and the separated time equation explicated. As to the separated radial dependence, it results in a pair of coupled ordinary differential equations. They are disentangled in an elementary way and recast into the form of an ordinary quantum like equation. The energy spectrum is then qualitatively obtained on the base of standard mathematical results concerning Schrödinger eigenvalue problem. The solution of the radial equations, is explicitly determined for flat, open and closed universe.

A. Zecca (✉)
Dipartimento di Fisica dell' Universita', Via Celoria 16, 20133 Milano, Italy
e-mail: Antonio.Zecca@mi.infn.it

A. Zecca
GNFM, Gruppo Nazionale per la Fisica Matematica, Sesto Fiorentino (Fi), Italy

A main result is that the solutions and the spectrum do not depend on the mass of the particle and coincide with the one previously studied in the massless neutrino case. The closed universe case is further studied by assimilating it to a closed quantum system for which standard boundary conditions on the eigenfunctions are required. The corresponding discrete spectrum is asymptotically determined in a special but meaningfull case. The result, however, is very from being verifiable by the present experimental data relative to the energy spectrum of cosmic spin $1/2$ particles.

2 Reduction of Dirac Equation to One Dimension Quantum-Like Equation

The Dirac field equation in curved space-time can be written [4, 6, 7] in terms of two coupled spinor equation in two spinor:

$$\begin{aligned} \nabla_{A\dot{X}} P^A + i\mu_\star \bar{Q}_{\dot{X}} = 0, \\ \nabla_{A\dot{X}} Q^A + i\mu_\star \bar{P}_{\dot{X}} = 0 \end{aligned} \tag{1}$$

with $\mu_\star\sqrt{2} = m_0$ the mass of the particle. Equation (1) has been solved in the Robertson-Walker (RW) metric [8]

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (a = 0, \pm 1) \tag{2}$$

by using the Neumann-Penrose formalism [9] and by variable separation [4]

$$\begin{aligned} (P^0, P^1) &\equiv \frac{e^{im\varphi}}{rR(t)}\big(\big[F(r)T(t) + G(r)S(t)\big]S_1,\ \big[F(r)T(t) - G(r)S(t)\big]S_2\big), \\ (\bar{Q}^1, -\bar{Q}^0) &\equiv \frac{e^{im\varphi}}{rR(t)}\big(\big[F(r)T(t) - G(r)S(t)\big]S_1,\ \big[F(r)T(t) + G(r)S(t)\big]S_2\big), \end{aligned} \tag{3}$$

$(S_1 = S_1(\theta),\ S_2 = S_2(\theta))$ where $m = 0, \pm 1, \pm 2, \ldots$. The results are the following. The separated angular equations have the solutions [5]:

$$\begin{aligned} S_{2lm} &= (1-\eta)^{\frac{m}{2}+\frac{1}{4}}(1+\eta)^{\frac{m}{2}-\frac{1}{4}} P_{l-m}^{(m+\frac{1}{2}, m-\frac{1}{2})}(\eta), \quad m \geq 1,\ l = m, m+1, \ldots, \\ S_{2lm} &= (1+\eta)^{\frac{|m|}{2}+\frac{1}{4}}(1-\eta)^{\frac{|m|}{2}-\frac{1}{4}} P_{l-|m|}^{(|m|-\frac{1}{2}, |m|+\frac{1}{2})}(\eta), \quad m \leq -1,\ l = |m|, |m|+1, \ldots, \\ S_{2l0} &= \sin\theta\, U_l(\eta), \quad \eta = \cos\theta, \end{aligned} \tag{4}$$

$P_n^{(\alpha,\beta)}$ the Jacobi polynomials [10]. (The explicit expression of the solution S_1 follows from that of S_2 by the substitution $m \to -m$.) By choosing $-ik$ (instead of k as done in [4]) to be the separation constant of radial and time dependence, one has $S(t) \cong T^\star(t)$ with T obeying

$$\ddot{T} + 2\frac{\dot{R}}{R}\dot{T} + \left(\frac{\ddot{R}}{2R} + \frac{1}{4}\frac{\dot{R}^2}{R^2} - im_0\sqrt{2}\frac{\dot{R}}{R} + 2m_0^2 + \frac{k^2}{R^2}\right)T = 0 \tag{5}$$

that explicitly depends on the time evolution of the cosmological background (examples of different cosmological evolutions are given in [4]).

Starting from (1) and using the same decomposition given by (3), one obtains for the radial equation the following expression:

$$\begin{aligned} r\sqrt{1-ar^2}G' - ikF &= \lambda G, \\ r\sqrt{1-ar^2}F' - ikG &= -\lambda F \end{aligned} \tag{6}$$

where $\lambda^2 = (l+\frac{1}{2})^2,\ l = |m|, |m|+1, \dots$ for $|m| \geq 1$ and $\lambda^2 = (l+1)^2,\ l = 0, 1, 2, \dots$ for $m = 0$ [5]. The equations can be decoupled by first setting

$$r_\star = \int_0^r \frac{dr}{\sqrt{1-ar^2}}, \quad a = 0, \pm 1 \tag{7}$$

so that they become

$$\begin{aligned} \left(\frac{d}{dr_\star} - \frac{\lambda}{r}\right)G &= ikF, \\ \left(\frac{d}{dr_\star} + \frac{\lambda}{r}\right)F &= ikG \end{aligned} \tag{8}$$

and then denoting $Z_+ = F$, $Z_- = G$

$$\begin{aligned} \left(\frac{d^2}{dr_\star^2} + k^2\right)Z_\pm &= V_\pm Z_\pm, \\ V_\pm &= \frac{\lambda^2}{r^2} \mp \frac{\lambda}{r^2}\frac{dr}{dr_\star}. \end{aligned} \tag{9}$$

These equations are the same as the neutrino massless case [5].

3 Discussion of the Radial Equation

According to standard results in the theory of one dimensional eigenvalue problems [11] it is possible to determine the nature of the spectrum of values k^2 of (9). For $a = 0$, (flat universe) $r_\star = r,\ r \geq 0$. From the analytical behavior of $V_\pm$ then k^2 may take any positive value. For $a = -1$ (open universe) $r_\star = \sinh^{-1},\ r_\star \geq 0$, the spectrum relative to V_+ is the positive real axis. That relative to V_- is again the real axis plus an a priori possible discrete set of negative eigenvalues so that the common spectrum is again $k^2 > 0$. For $a = 1$ (closed universe) $r_\star = \sin^{-1},\ 0 \leq r_\star \leq \frac{\pi}{2}$, there is a discrete positive spectrum for k^2. This is coherent with the interpretation of system in the closed universe as a confined quantum system. Equations (9) can be explicitly integrated.

For $a = 0$ the equation to be solved is

$$Z''_\pm + \left[k^2 - \frac{\lambda}{r^2}(\lambda \mp 1)\right]Z_\pm = 0 \tag{10}$$

that can be reported to the hypergeometric confluent equation. One obtains the solution

$$Z_+(r) = r^\lambda e^{ikr}\big[A\Phi(\lambda, 2\lambda; -2ikr) + B\Psi(\lambda,\ 2\lambda+1; -2ikr)\big] \tag{11}$$

where Ψ is the logarithmic confluent solution [10]. Similarly for Z_-.

If $a=1$, the equation to solve is

$$\left(-\frac{d^2}{dr_\star^2}+\frac{\lambda(\lambda\mp\cos r_\star)}{\sin^2 r_\star}\right)Z_\pm=k^2 Z_\pm. \tag{12}$$

By setting $\xi=\cos r_\star$ we transform (12) concerning Z_+ into

$$Z_+''+\frac{\xi}{\xi^2-1}Z_+'-\left[\frac{k^2}{\xi^2-1}+\frac{\lambda-\xi}{(\xi-1)^2(\xi+1)^2}\right]Z_+=0. \tag{13}$$

As to Z_-, one can proceed similarly.

Finally, if $a=-1$, the equation for Z_+ coincides with (13) relative to $a=1$ after the substitution $k\to ik$. Also here one can proceed similarly for Z_-.

Due to the interest, one can develop the case $a=1$ that must be solved under the conditions

$$Z_\pm(r_\star=0)=Z_\pm\left(r_\star=\frac{\pi}{2}\right)=0. \tag{14}$$

By further setting $Z_+=(1-\xi)^{\frac{\lambda}{2}}(1+\xi)^{\frac{\lambda+1}{2}}f(\xi)$ (13) becomes

$$(1-\xi^2)f''+\left[1-2(\lambda+1)\xi\right]f'-\left(\lambda^2+\lambda+\frac{1}{4}-k^2\right)f=0. \tag{15}$$

Its solution in terms of hypergeometric function [10] is

$$\begin{aligned}Z_+=(1-\xi)^{\frac{1-\lambda}{2}}(1+\xi)^{\frac{\lambda+1}{2}}\bigg[&AF\left(1-k,\,1+k;\lambda+\frac{3}{2};\frac{1+\xi}{2}\right)\\&+B\left(\frac{1+\xi}{2}\right)^{-\lambda-\frac{1}{2}}F\left(\frac{1}{2}-\lambda-k,\,\frac{1}{2}-\lambda+k;\frac{1}{2}-\lambda;\frac{1+\xi}{2}\right)\bigg].\end{aligned} \tag{16}$$

(The equation for Z_- follows from that of Z_+ with $\xi\to-\xi$.) The solution (16) generalizes the one obtained in [5] for $\lambda=1$. As shown in Ref. [5] the condition (14) for $\lambda=1$ implies the constraint $2k\tan(\pi k/2)=1$ that in turn gives a countable set of solutions k_n that for large n are of the form

$$k_n\cong 4n\quad (n\gg 1). \tag{17}$$

(With regards to this, notice that the spectrum $k_n\propto n$ proposed in [2] by a different argument, should be considered correct only for large n.) The condition $Z_+(0)=Z_+(\frac{\pi}{2})=0$ for the general solution (16) seems quite difficult to be exploited. (A hint to exploit (14) could be of iterating the procedure developed in [2] for $\lambda=1$ to obtain the constraint equation for k.) The equation for Z_- with the constraint (14) determines the same spectrum of values k_n^2 that, in the considered special case, represents the energy spectrum for the Dirac wave equation in the closed Robertson-Walker space-time.

4 Concluding Remarks

In the previous sections it has been shown that the study of the Dirac equation in RW metric can be reduced to the solution of one dimensional Schrödinger-like equation. The main consequence is that the spectrum of the Schrödinger operator results to be independent of the mass of the particle and coincides with the one previously determined for the massless neutrino case. In particular, in the closed universe case, the spectrum is, for $\lambda = 1$, of the form $E_n \equiv k_n^2 \cong (4n)^2$ for large n. According to this result for $\Delta E_n = E_{n+1} - E_n \simeq 1$ eV then $E_n \simeq 10^{26}$ eV and for $E_n = 4$ MeV then $\Delta E_n \simeq 10^{-10}$ eV. The separation of the energy levels is therefore beyond the present experimental sensitivity not only for neutrinos but also for galactic spin $1/2$ particles. Indeed an experimental separability of neutrino energy levels would involve very improbable energies. On the other hand, an high energy value seems not so improbable for spin $1/2$ galactic particles. Unfortunately the relative separation of the discrete levels obtained is completely negligible compared to the error $\Delta E/E \geq 10 - 30\%$ of the present experimental measurements [12–14].

The problem remains of interest in principle. An experimental evidence of the non existence of a discrete component in the energy spectrum of spin $1/2$ cosmic particles would rule out the idea of a closed universe in case of the standard cosmological model.

Acknowledgements It is a pleasure to thank Profs. M. Bersanelli, E. Meroni, B. Caccianiga and P. Rancoita for discussions and comments on the experimental data.

References

1. Barut, A.O., Duru, I.H.: Exact solution of the Dirac equation in spatially flat Robertson-Walker spacetimes. Phys. Rev. D **36**, 3705 (1987)
2. Zecca, A.: The Dirac equation in the Roberson-Walker space-time. J. Math. Phys. **37**, 874 (1996)
3. Huang, X.B.: Exact solutions of the Dirac equation in Roberson-Walker space-time. arXiv:gr-gq/0501077
4. Zecca, A.: Solution of the Dirac equation in expanding universes. Int. J. Theor. Phys. **45**, 44 (2006)
5. Montaldi, E., Zecca, A.: Neutrino wave equation in the Robertson-Walker geometry. Int. J. Theor. Phys. **33**, 1053 (1994)
6. Chandrasekhar, S.: The Mathematical Theory of Black Holes. Oxford University Press, New York (1983)
7. Penrose, R., Rindler, W.: Spinors and Space-Time. Cambridge University Press, Cambridge (1984)
8. Weinberg, S.: Gravitation and Cosmology. John Wiley Inc., New York (1972)
9. Newman, E.T., Penrose, R.: An approach to gravitational radiation by a method of spin coefficients. J. Math. Phys. **3**, 566 (1962)
10. Abramovitz, W., Stegun, I.E.: Handbook of Mathematical Functions. Dover, New York (1970)
11. Reed, M., Simon, B.: Functional Analysis. Academic Press, New York (1978)
12. Aguilar, M., et al.: Cosmic-ray positron fraction measurement from 1 to 30 GeV with AMS-01. Phys. Lett. B **646**, 145 (2007)
13. Abdo, A.A.: Measurement of the cosmic ray $e^+ + e^-$ spectrum from 20 GeV to 1 TeV with the Fermi large area telescope. Phys. Rev. Lett. **102**, 181101 (2009)
14. Di Bernardo, G., Gaggero, D., Grasso, D.: The high energy cosmic ray electron spectrum measured by Fermi gamma-ray space telescope: some possible interpretations. In: 1 Fermi Symposium, Washington, DC, Nov. 2–5, 2009